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ON 2-ABSORBING IDEALS OF COMMUTATIVE SEMRINGS

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Dedication

The second phase of my dreams comes true. Firstly, I would like to thank **God** for giving me guidance, strength, power of mind, and the moment to see my master's thesis.

To my parents, my sisters, my brother who are continuous to learn, love, and who has been a source of support and inspiration. I thankfully dedicate this thesis to you.

Last but not least, I am deeply dedicating this work to my teachers and especially professor Mohammed Saleh, Dr. Marwan Aloqeili and Dr. Khaled Adarbeh for superviing and giving me a continuous encouragement.

Declaration

I certify that this thesis, submitted for the degree of Master of Mathematics to the Department of Mathematics at Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis or any part of it has not been submitted for a higher degree to any other university or institution.

Leena Sawalmeh	Signature
June, 2019	....

Abstract

Suppose that S is a commutative semiring with unity different than zero. In this thesis, we study the concept of 2-absorbing ideal of S which can be considered as a generalization of prime ideals. We introduce some of its basic characteristics which are analogous to commutative ring theory and prove that the radical of 2-absorbing ideal is also 2-absorbing ideal and there are at most two prime k-ideals of S that are minimal over a 2-absorbing ideal. Moreover, we investigate the concept of 2-absorbing in special categories of semirings and prove some of advanced theorems related to it.

Keywords: Semiring, prime ideal, 2-absorbing ideal, divided semidomain.

الملخص

في الجزء الأول من هذه الرسالة، نقوم بدراسة المثاليات ثانوية الامتصاص في شبه الحلقات التبديلية و هي من أحد التعميمات المتعلقة بمفهوم المثاليات الأولية. ثانياً، نستكشف الخصائص الأساسية في هذا المفهوم بحيث أن هذه الخصائص تكون متقاربة في الحلقات التبديلية. أخيراً، نقوم بدراسة مفهوم المثاليات ثانوية الامتصاص في فصول خاصة في شبه الحلقات التبديلة.

كلمات مفتاحية : شبه الحلقات، المثاليات الأولية، المثاليات ثانوية الامتصاص، شبه المجال المقسم.

Contents

List of Symbols

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integer numbers
$\mathbb{R}$	Real numbers
S	Commutative semiring with unity
$\langle m \rangle$	The principal ideal generated by m.
$\langle n, m \rangle$	The ideal generated by n and m.
$S[x]$	The polynomial semiring over a semiring S.
$\mathbb{Z}_0^+(\mathbb{N})$	The set of non negative integers.
$(\mathbb{Z}_0^+(\mathbb{N}), +, \cdot)$	The semiring of all non negative integers under usual addition and multiplication.
$V(S)$	The set of elements of a semiring S having an additive inverse.
$U(S)$	The set of elements of a semiring S having a multiplicative inverse.
$Rad(I)$	The radical of an ideal I.
$Nil(S)$	The nilradical of a semiring S.
$Z(S)$	The set of all zero divisors of S.

CHAPTER 1

Introduction

The algebraic structure of semirings that are considered as a generalization of rings plays an important role in different branches of mathematics especially in applied science and computer engineering.

We assume throughout this thesis that all semirings are commutative with unity $1 \neq 0$. The first formal definition of semiring was introduced by H.S Vandiver in 1934 [18] and his paper entitled "*Note on a simple type of algebra in which cancelation law of addition does not hold*". This structure is known as "semiring".

In 1958, Henriksen [12] defined the special kind of ideals of a semiring which is called k-ideal or subtractive.

Prime ideals are essential appliance in semiring theory and many mathematicians have exploited the usefulness of the structure of prime ideals in algebraic systems over the decades. One of the generalizations of that concept is 2-absorbing ideals. In 2007, Badawi [6] introduced the concept of a 2-absorbing ideal of a commutative ring R with unity

$1 \neq 0$ and studied some of its basic properties. Badawi also proved that a proper nonzero ideal is a 2-absorbing ideal if and only If $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2 and I_3 of S, then either $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_1I_3 \subseteq I$.

In 2012, Darani [8] introduced the connotation of a 2-absorbing ideal of a commutative semiring. A nonzero proper ideal I of a semiring S is called a 2-absorbing ideal of S if whenever $x, y, z \in S$ with $xyz \in I$, then either $xy \in I$ or $xz \in I$ or $yz \in I$. In the same paper, a further generalization and some results corresponding to ring theory were also introduced.

In 2012, Ghaudhari [10] studied the 2-absorbing ideals in commutative semirings and introduced some of its properties in the quotient semiring and polynomial semiring.

Our research is organized as follows: in chapter (2), we recall some definitions, concepts and theorems in semiring theory. In chapter (3), we study and investigate the basic characteristics of the notation of 2-absorbing ideals which are analogous to ring theory. In chapter (4), we introduce the relation between P-primal ideals and 2-absorbing ideals and we also study the concept of 2-absorbing ideal in a divided semidomains and valuation semirings.

To be clear, all results I get in this thesis are generalization of the results obtained by Badawi [6].

CHAPTER 2

Preliminaries

In this chapter, we give basic information in semiring theory which are useful in the remainder of this thesis.

2.1 General Basics in semiring

In this section, we recall some basic concepts, definitions and theorems in semiring theory. Notice that, we assume throughout this thesis that S is a commutative semiring with unity $1 \neq 0$.

Definition 2.1 (Semigroup)**.** [11] A semigroup $(M, *)$ is an algebraic structure consisting of a nonempty set M together with an operation $*$ such that the following properties hold:

1. The operation $*$ is binary that is $a * b \in M$ for all $a, b \in M$.
2. The operation $*$ is associative that is $(a * b) * c = a * (b * c)$ for all a, b and $c \in M$.

Definition 2.2 (Monoid)**.** [11] A monoid $(M, *)$ is a semigroup with an identity element that means there exists an element e in M such that for every element $x \in M$ the equations $x * e = e * x = x$ hold. Therefore, the monoid is characterized by the triple $(M, *, e)$.

Definition 2.3. A commutative monoid (an abelian monoid) is a monoid whose operation is commutative.

Remark 2.1.
- *Any monoid is a semigroup.*
- *A semigroup $(M, *)$ can be embedded into a monoid by adding an identity element e not in M and defining $x * e = e * x = x$ for all $x \in M$.*

Example 2.1.
1. The set of positive integers $\mathbb{P}$ under addition is semigroup.
2. The set of square matrices over real numbers $\mathbb{R}$ under multiplication $(M_n(\mathbb{R}), \cdot)$ is a monoid with identity element is the identity matrix I.
3. The set of integers $\mathbb{Z}$ under multiplication is a commutative monoid with identity element is one.
4. The set of nonnegative integers $\mathbb{N}$ with addition form a commutative monoid with identity element is zero.

Definition 2.4 (Semiring)**.** [11] A semiring is an algebraic structure consists a nonempty set S with two binary operation addition $(+)$ and multiplication $(\cdot)$ such that the following are satisfied:

1. $(S, +)$ is a commutative monoid with identity element "0".
2. $(S, \cdot)$ is a monoid with identity element "1"
3. Left and right distribution laws hold, i.e. for all a, b and $c \in S$ we have:

- $a(b+c) = ab + ac$.
- $(b+c)a = ba + ca$.

4. $1 \neq 0$.

5. Multiplication by 0 annihilates S that is for all $s \in S$ we have:

   - $0s = s0 = 0$

Definition 2.5. A semiring S is said to be commutative if $ab = ba$ for all $a, b \in S$.

Example 2.2. The set of natural numbers $\mathbb{N}$ under usual addition and multiplication is a commutative semiring.

Example 2.3. Consider the triple $(\mathbb{N}, \oplus, \odot)$ of natural numbers $\mathbb{N}$ and $\oplus$ is defined by $x \oplus y$ is the least common multiple of x and y that is $(x \oplus y = lcm(x, y) = \frac{xy}{gcd(x,y)})$ and $\odot$ is the usual multiplication. Then $(\mathbb{N}, \oplus, \odot)$ is not semiring since the conditions $(1) - (3)$ are satisfied but (4) and (5) are not satisfied. To show that:

1. $(\mathbb{N}, \oplus)$ is a commutative monoid with identity element "1" since:

   - The associative property holds from number theorey, i.e., $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all a, b and $c \in \mathbb{N}$.
   - "1" is the additive identity element since $1 \oplus b = b \oplus 1 = b$ for all $b \in \mathbb{N}$.
   - The commutative property holds since $a \oplus b = lcm(a, b) = \frac{ab}{gcd(a,b)}$ for all $a, b \in \mathbb{N}$.

2. $(\mathbb{N}, \odot)$ is a monoid with identity element "1" since:

   - The associative property holds since $a \odot (b \odot c) = a(bc) = (ab)c = (a \odot b) \odot c$ for all a, b and $c \in \mathbb{N}$.
   - "1" is the multiplicative identity since $1 \odot b = b \odot 1 = b$ for all $b \in \mathbb{N}$.

3. Left and right distribution laws hold since:

   - Let a, b and $c \in \mathbb{N}$. Then
   $$a \odot (b \oplus c) = \frac{abc}{gcd(b, c)}$$
   and
   $$(a \odot b) \oplus (a \odot c) = \frac{abac}{gcd(ab, ac)} = \frac{abc}{gcd(a, b)}$$
   So, left distribution law is satisfied.

   - Similarly, as we are done above the right distribution law is satisfied.

4. The additive identity element is the same of multiplication.

5. "0" doesn't annihilates $\mathbb{N}$ since:

   - $1 \odot a = a \odot 1 = a \neq 1$

Example 2.4. Consider $S = \mathbb{N}[x]$ be the set of all polynomial with coefficients in $\mathbb{N}$ where x is an indeterminate. Let the usual addition and multiplication operations of polynomials be defined on S. Then $(S, +, \cdot)$ is a semiring and it is called **The polynomial semiring over the semiring** $(\mathbb{N}, +, \cdot)$.

Example 2.5. Consider $S = \mathbb{N} + nx\mathbb{N}[x]$ with usual addition and multipliaction operations where x is an indeterminate and $n \in \mathbb{N}$. Then $(\mathbb{N} + nx\mathbb{N}[x], +, \cdot)$ is a commutative semiring.

Proposition 2.1. *[11] Let S be a nonempty set with two binary operations "$+$" and "$\cdot$" and two distinct elements "0" and "1". Then S is a commutative semiring if and only if the following are satisfied for all a, b, c, d and $e \in S$:*

(1) $a + 0 = 0 + a = a$.

(2) $a \cdot 1 = a$.

(3) $0 \cdot a = 0$.

(4) $[(ae + b) + c]d = db + [a(ed) + cd]$.

Proof. $(\Rightarrow)$ If S is a commutative semiring, then the conditions $(1)-(4)$ are trivially satisfied.

$(\Leftarrow)$ If the conditions $(1)-(4)$ are satisfied, then we want to show $(S, +, \cdot)$ is a commutative semiring.

(1) $(S, +, \cdot)$ is a commutative monoid with identity element "0" since:

- " + " is commutative since if $a, b \in S$ then by condition (4) we have:

$$\begin{aligned} a + b &= [(a \cdot 1 + b) + 0] \cdot 1 \\ &= b + [a \cdot 1 + 0 \cdot 1] \\ &= b + a \end{aligned}$$

- " + " is associative since if a, b and $c \in S$ then by condition (4) we:

$$\begin{aligned} (a + b) + c &= (b + a) + c \\ &= [(b \cdot 1 + a) + c] \cdot 1 \\ &= 1 \cdot a + [b \cdot 1 + c] \\ &= a + (b + c) \end{aligned}$$

- "0" is the identity for the addition since if $a \in S$ then by condition (1) we have:

$$0 + a = a + 0 = a$$

(2) $(S, \cdot)$ is commutative monoid with an identity element "1" since:

- " $\cdot$ " is commutative since if $a, b \in S$ then by condition (4) we have:

$$\begin{aligned} ab &= [(0+a)+0] \cdot b \\ &= ba + [0 \cdot b + 0 \cdot b] \\ &= ba \end{aligned}$$

- " $\cdot$ " is associative since if a, b and $c \in S$ then by condition (4) we:

$$\begin{aligned} (ab)c &= [(ab+0)+0] \cdot c \\ &= c \cdot 0 + [a(bc) + 0 \cdot c] \\ &= a(bc) \end{aligned}$$

- "1" is the identity for the multiplication since if $a \in S$ then by condition (2) and the commutative property for multiplication we have:

$$a \cdot 1 = a = 1 \cdot a$$

(3) Left and right distribution hold since:

- Let a, b and $c \in S$. Then

$$\begin{aligned} (a+b)c &= [(a \cdot 1 + b) + 0] \cdot c \\ &= cb + [a(1 \cdot c) + 0 \cdot c] \\ &= cb + ac \\ &= ac + bc \end{aligned}$$

  Hence, the right distribution law is satisfied.

- Now the left distribution law holds since:

$$\begin{aligned} a(b+c) &= (b+c)a \\ &= ba + ca \\ &= ab + ac \end{aligned}$$

(4) $1 \neq 0$ from assumption.

(5) "0" annihilates S since if $a \in S$ then by condition (3) and the commutative property of multiplication we have:

$$a \cdot 0 = 0 \cdot a = 0$$

Therefore, $(S, +, \cdot)$ is a commutative semiring. □

2.2 Ideals

Ideals play a fundamental role in ring theory and semiring theory. During this section, we recall the connotation of ideals of semirings and we give some examples of it.

Definition 2.6 (Subsemiring)**.** A subsemiring U of a semiring $(S, +, \cdot)$ is a subset of S such that $(U, +, \cdot)$ is a semiring.

Proposition 2.2. *A subset U of a semiring S is a subsemiring if the following conditions hold:*

1. *0 and 1 belong to U.*

2. *U is closed under addition (i.e., $a + b \in U$for all $a,\ b \in U$).*

3. *U is closed under multiplication (i.e., $ab \in U$for all $a,\ b \in U$).*

Example 2.6. Let S be a semiring. Then $P(S) = \{s+1, s \in S\} \cup \{0\}$ is a subsemiring of S. Since

1. $P(S)$ is a subset of S.

2. $0 \in P(S)$ and $1 = 1 + 0 \in P(S)$.

3.
   - $P(S)$ is closed under addition since let $a,\ b \in P(S)$. If $a = b = 0$, then $a + b = 0 \in P(S)$. If $a = 0$ and $b \neq 0$, then there exists $s_1 \in S$ such that $b = s_1 + 1$ and thus $a + b = s_1 + 1 \in P(S)$. Now if $a,\ b \neq 0$, then there exist s_1 and s_2 such that $a = s_1 + 1$ and $b = s_2 + 1$ and thus $a + b = (s_1 + s_2 + 1) + 1 \in P(S)$.
   - Similarly, $P(S)$ is closed under multiplication.

Definition 2.7 (Ideal)**.** [11] An ideal I of a semiring S is a nonempty subset of S with the following conditions are satisfied:

(1) I is closed under addition (i.e., if $a,\ b \in I$, then $a + b \in I$).

(2) $SI \subseteq I$ (i.e., $sb \in I$ for all $s \in S$ and $b \in I$).

Definition 2.8. A proper ideal I of a semiring S is an ideal such that $I \neq S$ (i.e., $1 \notin I$).

Example 2.7. Let $(\mathbb{Z}, +, \cdot)$ be the semiring of integers with usual addition and multiplication. Suppose $I = \mathbb{N}$. Then $\mathbb{N}$ is subsemiring of $\mathbb{Z}$, but I is not an ideal since $-1 \cdot 2 = -2 \notin \mathbb{N}$.

Definition 2.9. The principle ideal generated by one element x in a semiring S is the multipliers of x, denoted by $\langle x \rangle$ or Sx.

Definition 2.10. Let a and b be elements of a semiring S. Then we define the ideal $\langle a, b \rangle$ to be the ideal generated by a and b (i.e., $\langle a, b \rangle = \{s_1 a + s_2 b \mid s_1, s_2 \in S\}$

Definition 2.11. Let S be a semiring and A and B be ideals of S. Then we define the addition and multiplication of ideals as follow:

- $A + B = \{a + b \mid a \in A, b \in B\}$.
- $A \cdot B = \{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \mid a_i \in A, b_i \in B, n \in \mathbb{N}\}$.

Proposition 2.3. *[14] Let S be a semiring and A, B and C be ideals of S. Then the follwoing statements are satisfied:*

(1) The sets $A + B$ and $A \cdot B$ are ideals of S.

(2) $A + (B + C) = (A + B) + C$ and $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

(3) $A + B = B + A$ and $A \cdot B = B \cdot A$.

(4) $A \cdot (B + C) = A \cdot B + A \cdot C$.

(5) If $A + B = \langle 0 \rangle$, then $A = B = \langle 0 \rangle$.

(6) $A + \langle 0 \rangle = A$, $A \cdot S = A$ and $A \cdot \langle 0 \rangle = \langle 0 \rangle$.

Definition 2.12 (k-Ideal)**.** [11] A subtractive ideal (k-ideal) I of a semiring S is an ideal such that if $x, x + y \in I$, then $y \in I$.

Definition 2.13. [11] An element a of a semiring S is an additively idempotent if $a+a = a$. The set of all additively idempotent is denoted by $I^+(S)$.

Definition 2.14. [11] An element a of a semiring S is a multiplicatively idempotent if $a^2 = a$. The set of all multiplicatively idempotent is denoted by $I^*(S)$.

Definition 2.15. [11] A semiring S is called an additively idempotent if $S = I^+(S)$.

Definition 2.16. [11] A semiring S is called a multiplicatively idempotent if $S = I^*(S)$.

Definition 2.17. [11] A semiring S is called an idempotent if $S = I^+(S) \cap I^*(S)$.

Example 2.8. Let $S = \{0, 1, d\}$ be the idempotent semiring so that $1 + d = d + 1 = d$. Then $\{0, d\}$ is an ideal of S but not subtractive.

Example 2.9. Let $\mathbb{N}$ be the semiring of natural numbers with usual addition and multiplication. Then $I = 3\mathbb{N}$ is subtractive ideal.

Remark 2.2.
- *The set of all elements in a semiring S having a multiplicative inverse is denoted by $U(S)$.*
- *The set of all elements in a semiring S having an additive inverse is denoted by $V(S)$.*
- *$V(S)$ is not empty since $0 \in V(S)$ and it is submonoid of $(S, +)$.*
- *If $x + y \in V(S)$, then x and y also belong to $V(S)$.*
- *S is ring if and only if $V(S) = S$.*

2.3 Prime, Maximal and Minimal Ideals

Throughout this section, we recall the definitions of prime, maximal and minimal ideals which are considered as the most important tool in this thesis.

Definition 2.18 (prime ideal)**.** [11] An ideal P of a semiring S is prime if whenever $HK \subseteq P$ for some ideals H and K, then either $H \subseteq P$ or $K \subseteq P$.

Definition 2.19. The set of all prime ideals of a semiring S is called the **spectrum** of S and is denoted by **Spec(S)**.

Remark 2.3.
- *Ang ring is a semiring.*
- *The set of all prime ideals of a ring R form a semiring with usual addition and multiplication of ideals.*

The following result is a generalization for the one in ring theory.

Proposition 2.4. *[11] Let S be a semiring and I an ideal of S. Then the following are equivalent:*

(1) I is a prime ideal.

(2) $\{xsy \mid s \in S\} \subseteq I$ if and only if $x \in I$ or $y \in I$.

(3) If $x, y \in S$ with $\langle x \rangle \langle y \rangle \subseteq I$, then either $x \in I$ or $y \in I$.

Corollary 2.1. *[11] Let S be a semiring and $x, y \in S$. Then the following conditions on a prime ideal I of S are equivalent:*

(1) If $xy \in I$, then either $x \in I$ or $y \in I$.

(2) If $xy \in I$, then $yx \in I$.

Proof. (1) $\Rightarrow$ (2). Assume (1) holds and let $xy \in I$. Then either $x \in I$ or $y \in I$. Since I is an ideal, then $yx \in I$.

(2) $\Rightarrow$ (1). Let $x, y \in S$ with $xy \in I$. Since I is an ideal, then $xys \in I$ for all $s \in S$. By (2), we conclude $ysx \in S$ for all $s \in S$. By proposition (2.4), we have either $x \in I$ or $y \in I$. $\square$

Corollary 2.2. *[11] Let S be a commutative semiring and I an ideal of S. Then I is a prime ideal if and only if $xy \in I$ implies that $x \in I$ or $y \in I$ for all $x, y \in S$.*

Proof. ($\Rightarrow$) Let S be a commutative semiring and I be a prime ideal. Assume $xy \in I$ and $H = \langle x \rangle$ and $K = \langle y \rangle$. We claim that $\langle xy \rangle = \langle x \rangle \langle y \rangle$. Let $a \in \langle xy \rangle$ implies that there exists $s \in S$ such that $a = xys = x(1)y(s) \in \langle x \rangle \langle y \rangle$. Now, let $a \in \langle x \rangle \langle y \rangle$. Then there exist $s_1, s_2 \in S$ such that $a = xs_1ys_2$. Since S is commutative, then we have $a = xys_1s_2 \in \langle xy \rangle$. Since $xy \in I$, then $HK \subseteq I$ and thus either $H = \langle x \rangle \subseteq I$ or $K = \langle y \rangle \subseteq I$. Hence, either $x \in I$ or $y \in I$.

($\Leftarrow$) Let H, K be ideals of S with $HK \subseteq I$ and $H \nsubseteq I$. Suppose $a \in H \setminus I$. Then for each $b \in K$ if $ab \in I$, then by assumption we have either $a \in I$ or $b \in I$. Since $a \notin I$, then for all $b \in K$ we have $b \in I$ and hence $K \subseteq I$. $\square$

Example 2.10. Consider the semiring $(\mathbb{N}, +, \cdot)$. Then the ideal $I = \mathbb{N} \setminus \{1\}$ is prime ideal. But, the set $I[t]$ of all polynomials with coefficients in I where t is an indeterminate is not prime ideal since $(3+t)(1+2t) = 3 + 7t + 2t^2 \in I[t]$ while neither $3 + t$ nor $1 + 2t$ belong to $I[t]$.

Definition 2.20 (Zero divisor)**.** [17] An element a of a semiring S is called a zero divisor if there exists $b \neq 0$ in S such that $ab = 0$. Moreover, the set of all zero divisors of S is denoted by $Z(S)$.

Example 2.11. Let $S = \mathbb{Z}_6$ be a semiring with an addition and multiplication operations modulo 6. Then 3 is a zero divisor since $3 \cdot 4 = 0$.

Example 2.12. Let $S = \mathbb{N} \times \mathbb{N}$ be a semiring with an addition operation defined as $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and a multiplication operation is defined as $(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 \cdot y_2)$. Then $(0, a)$ and $(b, 0)$ are zero divisors where $a, b \in \mathbb{N}$.

Definition 2.21 (Semidomain)**.** [4] Let S be a commutative semiring with unity $1 \neq 0$. Then S is said to be semidomain if $ab = 0$ implies that either $a = 0$ or $b = 0$ (i.e., S has no nonzero zero divisors).

Remark 2.4. *A commutative semiring S is semidomain if and only if $\langle 0 \rangle$ is a prime ideal.*

Example 2.13. The semirings $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are semidomains.

Definition 2.22. A semiring S is said to be multiplicatively cancellative if $xy = xz$ for some elements x, y and z of S, then $y = z$.

Definition 2.23 (Maximal ideal)**.** [9] Let M be a proper ideal of a semiring. Then M is said to be a maximal ideal of S if there is no an ideal I of S such that $M \subset I \subset S$.

Definition 2.24. [7] A partially ordered set (POSet) is a nonempty set O with relation, usually denoted by $\leq$, such that the following conditions hold for all $a, b, c \in O$,

1. $a \leq b$ and $b \leq a$, then $a = b$ (Antisymmetry property).
2. $a \leq a$ (Reflexive property).
3. $a \leq b$ and $b \leq c$, then $a \leq c$ (Transitive property).

Definition 2.25. A totally ordered set is a partially ordered set $(O, \leq)$ with connexity property i.e., for all $x, y \in O$ either $x \leq y$ or $y \leq x$.

Definition 2.26. A totally ordered commutative monoid (tomonoid) $(M, +, \cdot, \leq)$ is a commutaive monoid $(M, +, \cdot)$ such that $(M, \leq)$ is a totally ordered set and if $x \leq y$ implies that $x + z \leq y + z$ for any $z \in M$.

Definition 2.27. [14] A multiplicatively closed set (MC-set) is a subset U of a semiring S such that:

1. $1 \in U$ (i.e., U is not empty).
2. $xy \in U$ for all $x, y \in U$.

In other words, U is an MC-set if and only if it is a submonoid of $(S, \cdot)$.

Definition 2.28. [7] A **chain C** in a partially ordered set $(O, \leq)$ is a subset of O such that for every $a, b \in C$, either $a \leq b$ or $b \leq a$. An element u of O is an **upper bound** of C if for every element $a \in C$, $a \leq u$. An element m of O is a **maximal** element of the partially ordered set O, if whenever $m \leq a$, $a \in O$ then $m = a$.

Proposition 2.5 (Zorn's lemma)**.** *[5] If every chain C in a partially ordered set O has an upper bound in O then O has at least one maximal element.*

Lemma 2.1. *[14] Let S be a semiring. Then the maximal elements of the set of all ideals disjoint from an MC-set of S are prime ideals.*

Proof. Let S be a semiring and $U \subseteq S$ an MC-set. Let $\mathscr{C}$ be the set of all ideals disjoint from U. If $\{I_\alpha\}$ is a chain of ideals belonging to $\mathscr{C}$, then $\cup I_\alpha$ is also an ideal disjoint from U and an upper bound for the chain $\{I_\alpha\}$. Therefore, according to Zorn's Lemma, $\mathscr{C}$ has a maximal element. Let P be a maximal element of $\mathscr{C}$.

Now we prove that P is a prime ideal of S. Let $a \notin P$, $b \notin P$ and $ab \in P$. Then $P \subset P + (a)$ and $P \subset P + (b)$. This means that $P + (a)$ and $P + (b)$ are ideals of S that are not disjoint from U. So there exist $u_1, u_2 \in U$ such that $u_1 = p_1 + xa$ and $u_2 = p_2 + yb$ for some $p_1,\ p_2 \in P$ and $x,\ y \in S$. But $u_1u_2 = p_1p_2 + p_1yb + p_2xa + xyab$. Since $ab \in P$, then $u_1u_2 \in P$ which contradicts this fact that P is disjoint from U. Therefore $ab \notin P$ and P is a prime ideal of S. $\square$

Corollary 2.3. *Any semiring has at least one maximal ideal and all maximal ideals are prime ideals.*

Proof. Let S be a semiring and $U = \{1\}$. Then U is an MC-set and the set $\mathscr{C}$ of all ideals disjoint from U is the set of all proper ideals of S. According to Zorn's lemma, $\mathscr{C}$ has a maximal element say P. By lemma (2.1), we conclude P is prime ideal and hence all maximal ideals of S are prime ideals. $\square$

Definition 2.29 (Minimal ideals)**.** Let m be an ideal of a semiring S. Then m is said to be a minimal ideal of S if there is no ideal I of S

such that $I \subset m \subset S$. In other words, $\{0\}$ is the only ideal that is properly contained in I.

Example 2.14. In the semiring $\mathbb{Z}_{12}$, the ideals $\langle 6\rangle$ and $\langle 4\rangle$ are minimal ideals.

Definition 2.30. An ideal m of a semiring S is said to be minimal ideal over an ideal I if there is no ideal J of S such that $I \subset J \subset m$.

Now, we consider the concept of radical of an ideal which is used widely during this thesis.

Definition 2.31. [14] Let I be an ideal of a semiring S. Then the radical of I is the set of all $a \in S$ such that $a^n \in I$ for some $n > 0$ and is denoted by $Rad(I)$.

$$Rad(I) = \{a \in S :\ a^n \in I\ for\ some\ n \in \mathbb{N}\}$$

Example 2.15. Consider the semiring of natural numbers $\mathbb{N}$. Then $Rad(4\mathbb{N}) = 2\mathbb{N}$ and $Rad(5\mathbb{N}) = 5\mathbb{N}$. In general, $Rad(n\mathbb{N}) = r\mathbb{N}$ where r is the product of all distinct prime factors of n.

Theorem 2.1. *[14] Let P be a prime ideal of a semiring S and n a positive integer. Then $Rad(P^n) = P$.*

Proof. Let $x \in P$. Then $x^n \in P^n$ and so $P \subset Rad(P^n)$. Now assume $x \in Rad(P^n)$, then there exists $m \in \mathbb{N}$ such that $x^m \in P^n$. Since $P^n \subset P$ and P is a prime ideal, we have $x \in P$ and so $P = Rad(P^n)$. □

Theorem 2.2. *Let I be an ideal of a commutative semiring S. Then the radical of I is also an ideal.*

Proof. Let $a, b \in Rad(I)$. Then there exist $m, n \in N$ such that $a^n, b^m \in I$. To show $a + b \in Rad(I)$, we use binomial theorem for $(a + b)^{m+n-1}$

as follow:

$$(a+b)^{m+n-1} = \binom{m+n-1}{0} a^{m+n-1} b^0 + \cdots + \binom{m+n-1}{i} a^{m+n-1-i} b^i + \cdots + \binom{m+n-1}{m+n-1} a^0 b^{m+n-1}$$

So for each i, we have either $i \geq m$ or $m+n-1-i \geq n$ and thus each term $a^{m+n-1-i}b^i \in I$. Hence, $a+b \in Rad(I)$. Now let $s \in S$ and $x \in Rad(I)$. Then there exists $m \in N$ such that $a^m \in I$. Since S is commutative and I is an ideal, then $(sa)^m = s^m a^m \in I$ implies that $sa \in Rad(I)$. Therefore, $Rad(I)$ is an ideal.

□

Definition 2.32. An element x in a semiring S is called nilpotent if $x^n = 0$ for some $n > 0$.

Definition 2.33. The set of all nilpotent elements of a semiring S is called nilradical of S and is denoted by $Nil(S)$.

Theorem 2.3. *The nilradical of a commutative semiring S ($Nil(S)$) is an ideal.*

Proof. Let $a, b \in Nil(S)$. Then there exist $m, n \in N$ such that $a^n = b^m = 0$. To show $a+b \in Nil(S)$, we use binomial theorem for $(a+b)^{m+n-1}$ as follow:

$$(a+b)^{m+n-1} = \binom{m+n-1}{0} a^{m+n-1} b^0 + \cdots + \binom{m+n-1}{i} a^{m+n-1-i} b^i + \cdots + \binom{m+n-1}{m+n-1} a^0 b^{m+n-1}$$

So for each i, we have either $i \geq m$ or $m+n-1-i \geq n$ and thus each term $a^{m+n-1-i}b^i = 0$. Hence, $a+b \in Nil(S)$. Now let $s \in S$ and $x \in Nil(S)$. Then there exists $m \in N$ such that $a^m = 0$. Since S is commutative and I is an ideal, then $(sa)^m = s^m a^m = 0$ implies that $sa = 0$. Therefore, $Nil(S)$ is an ideal. □

Definition 2.34 (Quotient semiring)**.** [4] Let I be an ideal of a commutative semiring S. Then the quotient semiring of S by I is $R/I = \{s + I : s \in S\}$ and the binary operations $\oplus$ and $\odot$ defined as follows: $(s_1+I)\oplus(s_2+I) = (s_1+s_2)+I$ and $(s_1+I)\odot(s_2+I) = (s_1 \cdot s_2)+I$

Definition 2.35 (P-Primal Ideal)**.** Let S be a semiring and I an ideal of S. Then I is said to be P-primal ideal of S if $Z(S/I) = P/I$ for some a prime ideal P.

Definition 2.36 (Division semiring)**.** Let S ba semiring. Then S is said to be a division semiring if $U(S) = S \setminus \{0\}$.

Definition 2.37 (Semifield)**.** A semiring S is said to be semifield if it is a commutative division semiring.

CHAPTER 3

Basic Characteristics of 2-Absorbing Ideals of Commutative Semiring

3.1 The Concept of 2-Absorbing Ideals

In this section, we give the concept of 2-absorbing ideals of a commutative semiring S which can be considered as a generalization of prime ideals and we introduce some examples related to it.

Definition 3.1. [8] A nonzero proper ideal I of a semiring S is called a 2-**absorbing ideal** of S if whenever $x, y, z \in S$ with $xyz \in I$, then either $xy \in I$ or $xz \in I$ or $yz \in I$.

Example 3.1. Let S be the semiring of all non negative integers under usual addition and multiplication $(Z_0^+, +, \cdot)$. Then the principle ideal $\langle 3 \rangle$ is 2-absorbing ideal of Z_0^+. **To show** this let $a, b, c \in Z_0^+$ with

$abc \in \langle 3 \rangle$ which implies that $3|abc$. Since 3 a is prime number, then we have either $3|a$ or $3|b$ or $3|c$ and thus either $3|ab$ or $3|ac$ or $3|bc$. So, we have either $ab \in \langle 3 \rangle$ or $ac \in \langle 3 \rangle$ or $bc \in \langle 3 \rangle$. **In general**, the ideals of the form $\langle p \rangle$ are 2-absorbing ideals of $(Z_0^+, +, \cdot)$ where p is a prime number.

Example 3.2. Let S be the semiring of nonnegative integers with identity element ∞ where addition and multiplication operations defined as $a \oplus b = \max\{a, b\}$ and $a \odot b = \min\{a, b\}$. We denote S by $(Z_0^+ \cup \{\infty\}, \oplus, \odot)$. Then $I_t = \{0, 1, 2, 3, ..., t\}$ where $t \in Z_0^+$ is 2-absorbing ideal since if $a \odot b \odot c \in I_t$ for some $a, b, c \in S$, then $a \odot b \odot c = \min\{a, b, c\} = a \ or \ b \ or \ c$. Hence, either $a \odot b \in I_t$ or $a \odot c \in I_t$ or $b \odot c \in I_t$.

Remark 3.1. *Every prime ideal of a commutative semiring S is a 2-absorbing ideal of S. But the converse is not true.*

Proof. Let I be a prime ideal of S and let $a, b, c \in S$ with $abc \in I$. Since I is a prime ideal, then either $a \in I$ or $b \in I$ or $c \in I$ and thus either $ab \in I$ or $ac \in I$ or $bc \in I$. Hence, I a is 2-absorbing ideal of S.

To show that the converse is not true we consider the following example.

Example 3.3. In the semiring $(Z_0^+, +, \cdot)$, let $I = \langle 4, 5 \rangle$. Then $I = \{0, 4, 5, 8, 9, 10, 12, 13, 14, ...\} = Z_0^+ \setminus \{1, 2, 3, 6, 7, 11\}$ is 2-absorbing ideal not prime ideal. **To show** that assume $abc \in I$ for some $a, b, c \in Z_0^+$ and suppose neither $ab \in I$ nor $ac \in I$ nor $bc \in I$. Then ab, bc and $ac \in \{1, 2, 3, 6, 7, 11\}$ and the possible choices for a, b, c are one of them belongs to $\{1, 2, 3, 6, 7, 11\}$ and the others equal 1 or $a = 2, b = 3, c = 1$. So, in either all cases we get $abc \in \{1, 2, 3, 6, 7, 11\}$ and not belong to I, a contradiction. Hence, I is 2-absorbing ideal.

I is not prime ideal of Z_0^+ since $2 \,.\, 7 \in \langle 4, 5\rangle$ but neither $2 \in \langle 4, 5\rangle$ nor $7 \in \langle 4, 5\rangle$.

$\square$

3.2 Properties in Semiring Theory Corresponding to Ring Theory

In this section, we give some properties of semiring theory that are similar to some properties in ring theory which are useful in the proof of theorems in section (3.3).

Lemma 3.1. *Let $I \subseteq P$ be ideals of a semiring S with P a prime ideal. Then the following conditions are equivalent:*

(1) P is a minimal prime ideal of I.

(2) $S - P$ is a multiplicatively closed set which is maximal with respect to missing I i.e., maximal among multiplicatively closed sets that are disjoint from I.

(3) For each $x \in P$, there is $y \notin P$ and a nonnegative integer n such that $yx^n \in I$.

Proof. (1)$\Rightarrow$(2) Expand $S - P$ to a multiplicatively closed set U that is maximal with respect to missing I. Let Q be an ideal containing I that is maximal with respect to being disjoint from U. Then by Lemma (2.1), Q is a prime ideal. Note that Q is also disjoint from $S - P$ which implies that $Q \subseteq P$. Since P is a minimal prime ideal of I, then $P \subseteq Q$ and thus $P = Q$. Since $Q \cap U = \phi$, then $U \subseteq S - Q = S - P$ and so $U = S - P$.

(2) $\Rightarrow$ (3) Choose a nonzero $x \in P$ and let $U = \{yx^i| \ y \in S - P, \ i = 0, 1, 2, ...\}$. Then U is a multiplicatively closed set that properly contains $S - P$. Since $S - P$ is maximal with respect to missing I, then there is some $y \in S - P$ and a nonnegative integer n such that $yx^n \in I$.

(3) $\Rightarrow$ (1) Assume that $I \subset Q \subset P$ where Q is a prime ideal. Choose an element $x \in P - Q$, then there exists an element $y \notin P$ and a positive integer n such that $yx^n \in I \subset Q$. since $y \notin Q \subset P$ and Q is prime ideal, then $x \in Q$, a contradiction. So, $P = Q$ □

Proposition 3.1. *Let S be a semiring and I a k-ideal of S. Then the Radical of I ($Rad(I)$) is the intersection of all prime k-ideals containing I.*

Proof. let Q be the intersection of all prime k-ideals of S containing I. Show $Rad(I) = Q$.

($\Rightarrow$) Let $x \in Rad(I)$. Then there exists $n \in \mathbb{N}$ such that $x^n \in I$ which implies $x^n \in P$ for any prime k-ideal P containing I. Since P is prime, then $x \in P$ and so $Rad(I) \subseteq Q$.

($\Leftarrow$) By contradiction, suppose there is $y \in Q$ such that $y \notin Rad(I)$. That means for any natural number n, $y^n \notin I$. So, the set $U = \{1, y, y^2, y^3, ...\}$ is an MC- set that is disjoint from I, then we can expand I to an k-ideal J that is maximal with respect to the disjointness of U. By lemma 2.1, J is prime ideal of S. Since Q is contained in all prime k-ideals that containing I and $y \in Q$, then we have $y \in J$, a contradiction. Hence, $Q \subseteq Rad(I)$. □

3.3 More Characteristics of 2-Absorbing Ideals

In this section, we discover basic properties of 2-absorbing ideals of a commutative semiring S. We study and prove some of advanced theorems which are generalization of ones in ring theory.

Theorem 3.1. *Suppose that I is a 2-absorbing ideal of a semiring S. Then $Rad(I)$ is also a 2-absorbing ideal of S and $x^2 \in I$ for every $x \in Rad(I)$.*

Proof. First, we show $x^2 \in I$ for all $x \in Rad(I)$. Let $x \in Rad(I)$. Then there exists $n \in \mathbb{N}$ with $x^n \in I$. By induction, if $n = 1$, then $x \in I$ and thus $x^2 \in I$. Assume it is true for $n = k$ that means if $x^k \in I$, then $x^2 \in I$. Now, show for $n = k + 1$. Suppose $x^{k+1} \in I$. Since I is 2-absorbing ideal and $x^{k+1} = x^{k-1}xx$, we conclude either $x^k \in I$ or $x^2 \in I$. In either cases we have $x^2 \in I$.

Now, let $xyz \in Rad(I)$ for some x, y and z of S. Then by the first part of the proof above $(xyz)^2 \in I$. Since S is commutative semiring, then we have $(xyz)^2 = x^2y^2z^2$. Since I is 2-absorbing ideal of S, then either $x^2y^2 \in I$ or $x^2z^2 \in I$ or $y^2z^2 \in I$. Hence, we have either $xy \in Rad(I)$ or $xz \in Rad(I) or$ $yz \in Rad(I)$. Therefore, $Rad(I)$ is 2-absorbing ideal of S. □

The converse of theorem 3.1 is not true to show that we consider the following example.

Example 3.4. In the semiring $(Z_0^+, +, .)$, let $I = \langle 3, 5\rangle = \{0, 3, 5, 6, 8, 9, 10, 11, ...\} = Z_0^+ \setminus \{1, 2, 4, 7\}$. Then $Rad(I) = \{a \in Z_0^+ : \ a^n \in I \ for \ some \ n \in \mathbb{N}\} = Z_0^+ \setminus \{1\}$ and it's a 2-absorbing ideal of Z_0^+ since if $abc \in Rad(I)$ for some $a, b, c \in Z_0^+$, then $abc \neq 1$ and thus either a or b or c doesn't equal 1. Hence, either $ab \in Rad(I)$

or $ac \in Rad(I)$ or $bc \in Rad(I)$. However, I is not 2-absorbing ideal of Z_0^+ since $2 \cdot 2 \cdot 2 \in I$ but $2 \cdot 2 \notin I$.

Theorem 3.2. *Suppose I is a 2-absorbing ideal of a semiring S. Then there are at most two prime k-ideals of S that are minimal over I.*

Proof. Let I be a 2-absorbing ideal of S. Suppose that $J = \{P_i \mid P_i$ is a prime k-ideal of S that is minimal over $I\}$ and suppose that J has at least three elements. Let $P_1, P_2 \in J$ be two distinct prime k-ideals. Then there are $x_1 \in P_1 \setminus P_2$ and $x_2 \in P_2 \setminus P_1$. We **claim** that $x_1x_2 \in I$. Since $P_1, P_2 \in J$, then by lemma (3.1) there exist $c_2 \notin P_1$ and $c_1 \notin P_2$ such that $c_2x_1^n \in I$ and $c_1x_2^m \in I$ for some $n, m \in \mathbb{N} \setminus \{0\}$. Since $x_1, x_2 \notin P_1 \cap P_2$ and P_1, P_2 are prime ideals, then $x_1, x_2 \notin I$ and $x_1^l, x_2^l \notin P_1 \cap P_2$ for all $l \in \mathbb{N} \setminus \{0\}$ implies that $x_1^l, x_2^l \notin I$. Since $c_2x_1^n, c_1x_2^m \in I$ and $x_1^l, x_2^l \notin I$, then $c_2x_1, c_1x_2 \in I$ because I is 2-absorbing ideal. Since $x_1, x_2 \notin P_1 \cap P_2$ and $c_2x_1, c_1x_2 \in I \subseteq P_1 \cap P_2$, we conclude $c_2 \in P_2 \setminus P_1$ and $c_1 \in P_1 \setminus P_2$, and thus $c_1, c_2 \notin P_1 \cap P_2$. Since $c_2x_1, c_1x_2 \in I$ and I is an ideal, then we have $(c_1 + c_2)x_1x_2 \in I$ and so either $(c_1 + c_2)x_1 \in I$ or $(c_1 + c_2)x_2 \in I$ or $x_1x_2 \in I$. If $(c_1+c_2)x_1 \in I \subseteq P_1 \cap P_2$, then either $(c_1+c_2) \in P_2$ or $x_1 \in P_2$ because P_2 is prime ideal of S. But $x_1 \notin P_2$, so we conclude $(c_1+c_2) \in P_2$. Since P_2 is k-ideal of S and $c_2 \in P_2$, then we have $c_1 \in P_2$, a contradiction. So, $(c_1+c_2)x_1 \notin I$. If $(c_1+c_2)x_2 \in I \subseteq P_1 \cap P_2$, then either $(c_1+c_2) \in P_1$ or $x_2 \in P_1$ because P_1 is prime ideal of S. But $x_2 \notin P_1$, so we have $(c_1 + c_2) \in P_1$. Since P_1 is k-ideal of S and $c_1 \in P_1$, then we have $c_2 \in P_1$, a contradiction. So, $(c_1 + c_2)x_2 \notin I$. Hence $x_1x_2 \in I$.

Now, suppose there is a $P_3 \in J$ such that P_3 is neither P_1 nor P_2. Then there exist $y_1 \in P_1 \setminus (P_2 \cup P_3)$, $y_2 \in P_2 \setminus (P_1 \cup P_3)$ and $y_3 \in P_3 \setminus (P_1 \cup P_2)$. Using previous claim we conclude $y_1y_2 \in I \subseteq P_1 \cap P_2 \cap P_3$ implies that $y_1y_2 \in P_3$. Since P_3 is prime ideal, then either $y_1 \in P_3$ or $y_2 \in P_3$, a contradiction. Hence, J has at most two elements. $\square$

Theorem 3.3. *Let I be a 2-absorbing k-ideal of a semiring S. Then one of the following statments must hold:*

(1) $Rad(I) = P$ is a prime k-ideal of S such that $P^2 \subseteq I$.

(2) $Rad(I) = P_1 \cap P_2$, $P_1P_2 \subseteq I$, and $Rad(I)^2 \subseteq I$ where P_1, P_2 are the only distinct prime k-ideals of S that are minimal over I.

Proof. By proposition (3.1) and theorem (3.2), we conclude that either $Rad(I) = P$ is a prime k-ideal of S or $Rad(I) = P_1 \cap P_2$ where P_1, P_2 are the only distinct prime k-ideals of S that are minimal over I. Suppose $Rad(I) = P$ is a prime k-ideal of S. Let $x, y \in P$. Using theorem (3.1), we have $x^2, y^2 \in I$ and thus $x^2y + xy^2 = x(x + y)y \in I$. Since I is 2-absorbing ideal of S, then we have either $xy \in I$ or $(x + y)y \in I$ or $x(x+y) \in I$. If $xy \in I$, then we are done. If $x(x+y) = x^2+xy \in I$, then $xy \in I$ because I is k-ideal of S and $x^2 \in I$. If $(x + y)y = xy + y^2 \in I$, then $xy \in I$ because I is k-ideal of S and $y^2 \in I$. Hence, each case implies $xy \in I$ and thus $P^2 \subseteq I$.

Now, suppose that $Rad(I) = P_1 \cap P_2$ where P_1, P_2 are the only distinct prime k-ideals of S that are minimal over I. To prove $Rad(I)^2 \subseteq I$ we follow the same argument above. Let $x, y \in Rad(I)$. Then by theorem (3.1), we have $x^2, y^2 \in I$. Now, $x^2y + xy^2 = x(x + y)y \in I$. Since I is 2-absorbing ideal of S, then we have either $xy \in I$ or $(x + y)y \in I$ or $x(x + y) \in I$. If $xy \in I$, then we are done. If $x(x + y) = x^2 + xy \in I$, then $xy \in I$ because I is k-ideal of S and $x^2 \in I$. If $(x + y)y = xy + y^2 \in I$, then $xy \in I$ because I is k-ideal of S and $y^2 \in I$. Hence, each case implies $xy \in I$ and thus $Rad(I)^2 \subseteq I$. Now, we show $P_1P_2 \subseteq I$. Let $x_1 \in P_1$ and $x_2 \in P_2$. Then we have three cases for x_1 and x_2:

- **Case 1**: If $x_1 \in P_1 \setminus P_2$ and $x_2 \in P_2 \setminus P_1$, then $x_1x_2 \in I$ (by the proof of theorem (3.2)).

- **Case 2**: If $x_1 \in P_1 \cap P_2 = Rad(I)$ and $x_2 \in P_2 \setminus P_1$. Since P_1 and P_2 are distinct and minmal over I, then we can pick $y_1 \in P_1 \setminus P_2$. By the proof of theorem (3.2), we have $y_1x_2 \in I$. Since $y_1, x_1 \in P_1$, then $y_1 + x_1 \in P_1$. Moreover, $y_1 + x_1 \notin P_2$ Since if $y_1 + x_1 \in P_2$, then $y_1 \in P_2$ because P_2 is k-ideal and $x_1 \in P_2$, a contradiction. Now, by the proof of theorem (3.2), we have $(x_1 + y_1)x_2 = x_1x_2 + y_1x_2 \in I$. Since I is k-ideal and $y_1x_2 \in I$, then we conclude $x_1x_2 \in I$.

- **Case 3**: If $x_2 \in P_1 \cap P_2 = Rad(I)$ and $x_1 \in P_1 \setminus P_2$, then by similar argument for case (2) we get $x_1x_2 \in I$.

Hence, in three cases above we have $x_1x_2 \in I$ and thus $P_1P_2 \in I$. □

Theorem 3.4. *Suppose that I is a 2-absorbing k-ideal of a semiring S and $Rad(I) = P$ is prime k-ideal such that $I \neq Rad(I)$. For each $a \in Rad(I) \setminus I$, let $B_a = \{s \in S \mid sa \in I\}$. Then B_a is a prime ideal of S so that $P \subseteq B_a$. Moreover, for all $x, y \in Rad(I) \setminus I$ either $B_x \subseteq B_y$ or $B_y \subseteq B_x$.*

Proof. Firstly, we show that $P \subseteq B_x$ for all $x \in P \setminus I$. Let $x \in P \setminus I$ and $y \in P$. If $y \in I$, then $yx \in I$ implies that $y \in B_x$. If $y \in P \setminus I$, then by theorem (3.3) $P^2 \subseteq I$ which implies $yx \in I$ and $y \in B_x$.

Secondly, we show that B_x is a prime ideal of S. Let $yz \in B_x$ for some $y, z \in S$. If $yz \in P$, then either $y \in P \subseteq B_x$ or $z \in P \subseteq B_x$ because P is prime ideal. If $yz \in B_x \setminus P$, then $yzx \in I$. Since $I \subseteq P$ and $yz \notin P$, we have $yz \notin I$. Since I is 2-absorbing ideal and $yz \notin I$, we have either $yx \in I$ or $zx \in I$ that means either $y \in B_x$ or $z \in B_x$.

Now, let $x, y \in P \setminus I$ and suppose that $z \in B_x \setminus B_y$. Since $P \subseteq B_y$, then $z \in B_x \setminus P$. We show $B_y \subset B_x$. Let $w \in B_y$. Then we have two cases for w:

- **Case 1**: If $w \in P$, then $w \in B_x$ because $P \subseteq B_x$.

- **Case 2**: If $w \in B_y \setminus P$, then $wy \in I$. Since $w \in B_y$ and $z \in B_x$ and I is an ideal, then we have $z(x+y)w \in I$. Since I is a 2-absorbing ideal, then we conclude that either $z(x+y) \in I$ or $zw \in I$ or $(x+y)w \in I$. If $z(x+y) \in I$, then $zy \in I$ because $zx \in I$ and I is k-ideal, a contradiction since $z \notin B_y$. So, $z(x+y) \notin I$. If $wz \in I \subseteq P$, then either $w \in P$ or $z \in P$ because P is a prime ideal, but neither $w \in P$ nor $z \in P$ so we have a contradiction and then $wz \notin I$. Therefore, $(x+y)w \in I$. Since I is a k-ideal and $yw \in I$, then $xw \in I$. Since S is a commutative semiring, then $xw = wx \in I$ and thus $w \in B_x$. Therefore, $B_y \subseteq B_x$.

□

Theorem 3.5. *Suppose that I is a 2-absorbing k-ideal of a semiring S and $Rad(I) = P_1 \cap P_2$ where P_1, P_2 are the only prime k-ideals of S that are minimal over I such that $P_1 \neq P_2$. Let $I \neq Rad(I)$. Then for each $a \in Rad(I) \setminus I$, $B_a = \{s \in S \mid sa \in I\}$ is a prime ideal of S such that $P_1 \cup P_2 \subseteq B_a$. Moreover, for all $x, y \in Rad(I) \setminus I$ either $B_x \subseteq B_y$ or $B_y \subseteq B_x$*

Proof. Firstly, we show that P_1, $P_2 \subset B_x$ for every $x \in Rad(I) \setminus I$. Let $x \in Rad(I) \setminus I$ and $y \in P_1$. If $y \in I$, then $yx \in I$ implies that $y \in B_x$. If $y \in P_1 \setminus I$, then by theorem (3.3) we have $P_1P_2 \subseteq I$. Since $x \in Rad(I)$ and $y \in P_1$, then $yx \in I$ implies that $y \in B_x$. Therefore, $P_1 \subset B_x$. A similar way we prove $P_2 \subset B_x$ for all $x \in Rad(I) \setminus I$.

Secondly, we show that B_x is a prime ideal of S. Let $yz \in B_x$ for some $y, z \in S$. If $yz \in P_1$, then either $y \in P_1 \subset B_x$ or $z \in P_1 \subset B_x$ because P_1 is prime ideal. If $yz \in P_2$, then either $y \in P_2 \subset B_x$ or $z \in P_2 \subset B_x$ because P_2 is prime ideal. Now, assume $yz \in B_x \setminus (P_1 \cup P_2)$. Then $yzx \in I$. Since $I \subset P_1 \cap P_2$ and $yz \notin P_1 \cup P_2$, then $yz \notin I$. Since

I is a 2-absorbing ideal of S and $yz \notin I$, then either $yx \in I$ or $zx \in I$, and thus either $y \in B_x$ or $z \in B_x$. Hence, B_x is a prime ideal of S.

Now, let $x, y \in Rad(I) \setminus I$ and $z \in B_x \setminus B_y$. Since $P_1, P_2 \subset B_y$, then $z \in B_x \setminus (P_1 \cup P_2)$. We show that $B_y \subset B_x$. Let $w \in B_y$. Then we have three cases for w:

- **Case1**: If $w \in P_1$, then $w \in B_x$ because $P_1 \subset B_x$.

- **Case2**: If $w \in P_2$, then $w \in B_x$ because $P_2 \subset B_x$.

- **Case3**: If $w \in B_y \setminus (P_1 \cup P_2)$, then $yw \in I$. Since $w \in B_y$, $z \in B_x$ and I is an ideal, then we conclude $(x + y)zw \in I$. Since I is a 2-absorbing ideal, then we have either $(x + y)z \in I$ or $(x + y)w \in I$ or $zw \in I$. We **claim** that $(x + y)w \in I$ since if $zw \in I \subset P_1 \cap P_2$, then $zw \in P_1$ and $zw \in P_2$. Since P_1 and P_2 are prime ideals of S, then we have either $z \in P_1$ or $w \in P_1$ and either $z \in P_2$ or $w \in P_2$, a contradiction because neither z nor w belong to $P_1 \cup P_2$. If $(x + y)z \in I$, then $yz \in I$ since I is k-ideal and $z \in B_x$, a contradiction because $z \notin B_y$. So, we have $(x + y)w = xw + yw \in I$. Since I is a k-ideal and $yw \in I$, then we get $xw \in I$. Since S is a commutative semiring, then $xw = wx \in I$ implies that $w \in B_x$. Therefore, $B_y \subset B_x$.

□

Corollary 3.1. *Assume that I is a 2-absorbing k-ideal of a semiring S and $G = \bigcup_{x \in Rad(I) \setminus I} B_x$ such that $I \neq Rad(I)$. Then I is a G-primal ideal of S.*

Proof. Suppose that I is a 2-absorbing ideal of S such that $I \neq Rad(I)$ and $G = \bigcup_{x \in Rad(I) \setminus I} B_x$. First, we want to show G is a prime ideal of S containing I. **To prove G is an ideal** let $a, b \in G$. Since $I \neq Rad(I)$, then there exist $x, y \in Rad(I) \setminus I$ such that $a \in B_x$ and $b \in B_y$. Since either $B_y \subseteq B_x$ or $B_x \subseteq B_y$ by theorems (3.4+3.5), then we have either $a, b \in B_x$ or $a, b \in B_y$. Since B_x and B_y are ideals of S by theorems $(3.4 + 3.5)$, then we have either $a + b \in B_x \subseteq G$ or $a + b \in B_y \subseteq G$ an hence $a + b \in G$. Now, let $s \in S$ and $g \in G$. Since $I \neq Rad(I)$, then there exists $x \in Rad(I) \setminus I$ such that $g \in B_x$. Since B_x is an ideal of S, then $sg \in B_x \subseteq G$ and thus G is an ideal of S. **To prove G is a prime ideal**, let $ab \in G$ for some $a, b \in S$. Since $I \neq Rad(I)$, then there exists $x \in Rad(I) \setminus I$ such that $ab \in B_x$. Since B_x is prime ideal by theorems $(3.4 + 3.5)$, then we have either $a \in B_x$ or $b \in B_x$. Since $B_x \subseteq G$, then we have either $a \in G$ or $b \in G$. **To prove I is contained in G**. Since for every $x \in Rad(I) \setminus I$, $Rad(I) \subseteq B_x$ by theorems $(3.4 + 3.5)$ and $I \subseteq Rad(I)$ by proposition (3.1), then $I \subseteq B_x \subseteq G$.

Now, we show $Z(S/I) = G/I$ where $G = \bigcup_{x \in Rad(I) \setminus I} B_x$.

($\Leftarrow$) Let $a + I \in G/I$. Then there exists $x \in Rad(I) \setminus I$ such that $a \in B_x$ which implies $ax \in I$. So, $ax + I = (a + I)(x + I) = I$ and thus $a + I$ is a zero divisor of S/I. Hence, we have $a + I \in Z(S/I)$.

($\Rightarrow$) Let $0 \neq a + I \in Z(S/I)$. Then there exists $0 \neq b + I \in S/I$ such that $(a + I)(b + I) = ab + I = I$. So, we have $a, b \notin I$ and $ab \in I$. We show $a, b \in G$ that means $a, b \in B_f$ for some $f \in Rad(I) \setminus I$. By theorem (3.3), we conclude that either $Rad(I) = P$ is a prime k-ideal of S or $Rad(I) = P_1 \cap P_2$ where P_1, P_2 are the only distinct prime k-ideals of S that are minimal over I.

- **Case 1**: Suppose $Rad(I) = P$ is a prime k-ideal of S. Since $ab \in I \subseteq P$ and P is prime ideal, then we have either $a \in P$ or $b \in P$. Since $a, b \notin I$, then we conclude either $a \in P \setminus I$ or $b \in P \setminus I$. If $a \in P \setminus I$, then $a \in B_a$ because $a^2 \in I$ (by theorem 3.1). Since $ab \in I$, then $b \in B_a$. If $b \in P \setminus I$, then $b^2 \in I$ by theorem

(3.1), which implies that $b \in B_b$. Also since $ab \in I$, then $a \in B_b$. Therefore, in either two cases we have $a, b \in G$ and thus in this case $Z(S/I) \subseteq G/I$.

- **Case 2**: Suppose that $Rad(I) = P_1 \cap P_2$ where P_1, P_2 are the only distinct prime k-ideals of S that are minimal over I. Since $ab \in I \subseteq Rad(I) = P_1 \cap P_2$, P_1 and P_2 are prime ideals, and $a, b \notin I$, then we have either $a \in P_1 \setminus I$ or $b \in P_1 \setminus I$ and either $a \in P_2 \setminus I$ or $b \in P_2 \setminus I$. So, we conclude either $a \in Rad(I) \setminus I$ or $b \in Rad(I) \setminus I$ or $b \in P_2 \setminus P_1$ and $a \in P_1 \setminus P_2$ or $b \in P_1 \setminus P_2$ and $a \in P_2 \setminus P_1$. Suppose $a \in Rad(I) \setminus I$. Then $a \in B_a$ since $a^2 \in I$ by theorem (3.1). Since $ab \in I$, $b \in B_a$ and so $a, b \in G$. Using similar argument we follow for the case if $b \in Rad(I) \setminus I$. Now, suppose $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$. Since $I \neq Rad(I)$, then there exists $d \in Rad(I) \setminus I$. Since $P_1 \subset B_d$ and $P_2 \subset B_d$ by theorem (3.5), we have $a \in B_d$ and $b \in B_d$ and so $a, b \in G$. Using similar argument we proceed for the case if $a \in P_2 \setminus P_1$ and $b \in P_1 \setminus P_2$. Therefore, in all cases we have $a, b \in G$ and thus $Z(S/I) \subseteq G/I$.

□

Theorem 3.6. *Assume that I is a k-ideal of a semiring S and suppose $Rad(I) = P$ is a prime k-ideal of S such that $I \neq Rad(I)$. Then the following statements are equivalent:*

(1) $B_a = \{s \in S \mid sa \in I\}$ is a prime ideal of S for each $a \in Rad(I) \setminus I$.

(2) I is a 2-absorbing ideal of S.

Proof. (2) $\Rightarrow$ (1) It follows from theorem (3.4).

(1) $\Rightarrow$ (2) Let $xyz \in I$ for some x, y and $z \in S$. Since $I \subset Rad(I)$ and $Rad(I) = P$ is a prime k-ideal of S, we have either $x \in Rad(I)$ or

$yz \in Rad(I)$. Suppose $x \in Rad(I)$. If $x \in I$, then $yx \in I$ and we are done. If $x \in Rad(I) \setminus I$, then $yz \in B_x$. Since B_x is a prime ideal, then we have either $y \in B_x$ or $z \in B_x$ and thus either $yx \in I$ or $zx \in I$. Now, assume $yz \in Rad(I)$. Since $Rad(I)$ is prime ideal of S, then we have either $y \in Rad(I)$ or $z \in Rad(I)$. So, in either case we proceed as in the case $x \in Rad(I)$. Hence, I a is 2-absorbing ideal of S. $\square$

Theorem 3.7. *Suppose that I is a k-ideal of a semiring S and assume $Rad(I) = P_1 \cap P_2$ where P_1, P_2 are the only distinct prime k-ideals of S that are minimal over I such that $I \neq Rad(I)$. Then the following statements are equivalent:*

(1) I is a 2-absorbing ideal of S.

(2) For each $a \in (P_1 \cap P_2) \setminus I$, $B_a = \{s \in S \mid sa \in I\}$ is a prime ideal of S and $P_1P_2 \subseteq I$.

(3) For each $a \in (P_1 \cup P_2) \setminus I$, $B_a = \{s \in S \mid sa \in I\}$ is a prime ideal of S.

Proof. (1) $\Rightarrow$ (2) It follows from theorems $(3.3 + 3.5)$.

(2) $\Rightarrow$ (3) Let $x \in (P_1 \cup P_2) \setminus I$. Then we have three cases for x:

- **Case 1**: If $x \in (P_1 \cap P_2) \setminus I$, then we are done by (2).

- **Case 2**: If $x \in P_1 \setminus (P_2 \cup I)$, We claim that $y \in P_2$ if and only if $yx \in I$ where $y \in S$. To show that if $y \in P_2$, then $yx \in I$ because $P_1P_2 \subseteq I$ and $x \in P_1$. Now if $yx \in I \subseteq P_1 \cap P_2$, then $yx \in P_2$. Since P_2 is a prime ideal of S and $x \notin P_2$, then we have $y \in P_2$. By the previous claim, we conclude $B_x = \{y \in S \mid yx \in I\} = P_2$ and thus it's a prime ideal of S.

- **Case 3**: If $x \in P_2 \setminus (P_1 \cup I)$, then using the same argument in **case (2)** we conclude that $B_x = P_1$ and thus it's a prime ideal of S.

(3) $\Rightarrow$ (1) Let $xyz \in I$ for some x, y and $z \in S$. Since $I \subseteq P_1 \cap P_2$ and P_1, P_2 are prime ideals of S, then we have either $x \in P_1$ or $y \in P_1$ or $z \in P_1$ and either $x \in P_2$ or $y \in P_2$ or $z \in P_2$. So we have three cases either $x \in P_1 \cup P_2$ or $y \in P_1 \cup P_2$ or $z \in P_1 \cup P_2$. **Without loss of generality**, assume $x \in P_1 \cup P_2$. Now, if $x \in I$, then $yx \in I$ and we are done. Otherwise if $x \in (P_1 \cup P_2) \setminus I$, then either $y \in B_x$ or $z \in B_x$ since $yz \in B_x$ and B_x is a prime ideal of S by (3). Thus, either $yx \in I$ or $zx \in I$ and so I is 2-absorbing ideal of S. □

Theorem 3.8. *Suppose that I is a 2-absorbing k-ideal of a semiring S and $I \neq Rad(I)$. For each $a \in Rad(I) \setminus I$, let $B_a = \{s \in S \mid sa \in I\}$. Then:*

(1) If $x \in Rad(I) \setminus I$ and $y \notin B_x$, then $B_{yx} = B_x$.

(2) If $x, y \in Rad(I) \setminus I$ and B_x is properly contained in B_y, then $B_{dx+qy} = B_x$ for every $q, d \in S$ such that $qd \notin B_x$. Moreover, if $x, y \in Rad(I) \setminus I$ and $B_x \subset B_y$, then $B_{x+y} = B_x$.

Proof. (1) Let $x \in Rad(I) \setminus I$ and $y \in S$ such that $yx \notin I$. Since $x \in Rad(I) \setminus I$, $yx \notin I$ and $Rad(I)$ is an ideal, then $yx \in Rad(I) \setminus I$ and so B_{yx} is defined. Now we show $B_{yx} = B_x$.

($\Leftarrow$) Let $z \in B_x$. Then $zx \in I$. Since I is an ideal, S is commutative semiring and $y \in S$, then we conclude $zyx \in I$ and thus $z \in B_{yx}$.

($\Rightarrow$) Let $c \in B_{yx}$. Then $cyx \in I$ and thus $yc \in B_x$. Since B_x is prime ideal by theorems $(3.4 + 3.5)$, then either $y \in B_x$ or $c \in B_x$. If $y \in B_x$, then $yx \in I$, a contradiction. Hence, $c \in B_x$ and thus $B_{yx} = B_x$.

(2) Let $x, y \in Rad(I) \setminus I$ and B_x is properly contained in B_y. Suppose $q, d \in S$ such that $qd \notin B_x$. Since B_x is a prime ideal of S,

then neither q nor d are in B_x. Since $x, y \in Rad(I)$ and $Rad(I)$ is an ideal, then we conclude $dx + qy \in Rad(I)$. Since B_x is a prime ideal containing $Rad(I)$ by theorems $(3.4 + 3.5)$ and $I \subseteq Rad_(I)$, then $q, d \notin I$. Hence, $dx + qy \in Rad(I) \setminus I$ and B_{dx+qy} is defined. Now we show $B_{dx+qy} = B_x$.

($\Leftarrow$) Let $c \in B_x$. Then $cx \in I$. Since $B_x \subset B_y$, then $c \in B_y$ and so $cy \in I$. Since I is an ideal, then $cdx + cqy \in I$ and thus $c \in B_{dx+qy}$. Hence, $B_x \subseteq B_{dx+qy}$.

($\Rightarrow$) Suppose $B_x \neq B_{dx+qy}$. Then B_x is properly contained in B_{dx+qy} i.e., $B_x \subset B_{dx+qy}$. By theorems $(3.4 + 3.5)$, we have two cases either $B_y \subseteq B_{dx+qy}$ or $B_{dx+qy} \subseteq B_y$. Since $B_x \subset B_y$ and $B_x \subset B_{dx+qy}$, then in both cases above we can find $z \in B_y \cap B_{dx+qy}$ such that $z \notin B_x$ that is $zy \in I$ and $z(dx+qy) \in I$. Since I is a k-ideal and $zqy \in I$, then $zdx \in I$ and thus $zd \in B_x$. Since B_x is a prime ideal of S, then either $z \in B_x$ or $d \in B_x$, a contradiction since neither $z \in B_x$ nor $d \in B_x$. Therefore, $B_x = B_{dx+qy}$.

Now, we show the last part of the theorem. Let $x, y \in Rad(I) \setminus I$. Since S is a semiring with unity 1, then we can take $q = d = 1$ and so $qd = 1 \notin B_x$ since if $1 \in B_x$ then $x \in I$, a contradiction. So, by second part of the theorem we have $B_{x+y} = B_x$. □

Theorem 3.9. *Let I be a nonzero k-ideal of a semiring S. Then the following statements are equivalent:*

1 I is a 2-absorbing ideal of S.

2 if $I_1 I_2 I_3 \subseteq I$, then either $I_1 I_2 \subseteq I$ or $I_2 I_3 \subseteq I$ or $I_1 I_3 \subseteq I$ where I_1, I_2 and I_3 are ideals of S.

Proof. (2) $\Rightarrow$ (1) Let $abc \in I$ for some $a, b, c \in S$. Then we **claim** that $\langle abc \rangle = \langle a \rangle \langle b \rangle \langle c \rangle$. To show that assume that $H = \langle a \rangle$,

$K = \langle b \rangle$ and $L = \langle c \rangle$ and let $x \in \langle abc \rangle$. Then $x = (abc)s = a(1)\, b(1)\, c(s) \in \langle a \rangle \langle b \rangle \langle c \rangle$ for some $s \in S$. Hence, $\langle abc \rangle \subseteq \langle a \rangle \langle b \rangle \langle c \rangle$. Now, let $x \in \langle a \rangle \langle b \rangle \langle c \rangle$. Then there exist s_1, s_2 and s_3 in S such that $x = (as_1)(bs_2)(cs_3)$. Since S is commutaive semiring, then $x = abc(s_1s_2s_3)$ which implies $x \in \langle abc \rangle$. Therefore $\langle abc \rangle = \langle a \rangle \langle b \rangle \langle c \rangle$. Since $abc \in I$ and $\langle abc \rangle = HKL$, then we have $HKL \subseteq I$. By assumption we have either $HK = \langle a \rangle \langle b \rangle \subseteq I$ or $KL = \langle b \rangle \langle c \rangle \subseteq I$ or $HL = \langle a \rangle \langle c \rangle \subseteq I$ and thus either $ab \in I$ or $bc \in I$ or $ac \in I$. Therefore, I is 2-absorbing ideal of S.

(1) $\Rightarrow$ (2) Suppose I is a 2-absorbing ideal of S and suppose that $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2 and I_3 of S. By theorem (3.3), we conclude that either $Rad(I) = P$ is a prime k-ideal of S or $Rad(I) = P_1 \cap P_2$ where P_1, P_2 are the only distinct prime k-ideals of S that are minimal over I. Assume $I = Rad(I)$, then either $I = P$ is a prime k-ideal of S or $I = P_1 \cap P_2$, where P_1, P_2 are the only distinct prime k-ideals of S that are minimal over I. If $I = P$ is a prime ideal, then by corollary (2.2) we have either $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$. **Without loss of generality**, assume $I_1 \subseteq I$ then $I_1I_2 \subseteq I_1 \subseteq I$ and $I_1I_3 \subseteq I_1 \subseteq I$. Now if $I = P_1 \cap P_2$, then we have $I_1I_2I_3 \subseteq P_1$ and $I_1I_2I_3 \subseteq P_2$. Since P_1 and P_2 are prime ideals of S, then we conclude either $I_1 \subseteq P_1$ or $I_2 \subseteq P_1$ or $I_3 \subseteq P_1$ and either $I_1 \subseteq P_2$ or $I_2 \subseteq P_2$ or $I_3 \subseteq P_2$. Assume $I_1 \subseteq P_1$. If $I_1 \subseteq P_2$, then $I_1 \subseteq P_1 \cap P_2 = I$ which implies $I_1I_2 \subseteq I$ and $I_1I_3 \subseteq I$ and we are done. If $I_1 \nsubseteq P_2$, then either $I_2 \subseteq P_2$ or $I_3 \subseteq P_2$. Since $P_1P_2 \subseteq I$ by theorem (3.5), we conclude that either $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$. Hence, the assumption holds for $I = Rad(I)$.

Now, suppose $I \neq Rad(I)$. We consider two cases:

- **Case 1**: Suppose that $Rad(I) = P$ is a prime ideal of S. Since $I_1I_2I_3 \subseteq I \subseteq P$ and P is a prime ideal of S, then we conclude either $I_1 \subseteq P$ or $I_2 \subseteq P$ or $I_3 \subseteq P$. **Without loss of generality**, assume $I_1 \subseteq P$. If $I_1 \subseteq I$, then $I_1I_2 \subseteq I_1 \subseteq I$ and we are done. Now suppose $I_1 \subseteq P$ and $I_1 \nsubseteq I$ and let $x \in I_1 \setminus I$. Since $xI_2I_3 \subseteq I$, we have $xab \in I$ for every $a \in I_2$ and $b \in I_3$ which implies that $I_2I_3 \subseteq B_x$. Since B_x is prime ideal of S by theorem (3.4), we conclude that either $I_2 \subseteq B_x$ or $I_3 \subseteq B_x$. we consider two cases for the previous conclusion:

- If $I_2 \subseteq B_x$ and $I_3 \subseteq B_x$ for all $x \in I_1 \setminus I$, then $xI_2 \subseteq I$ and $xI_3 \subseteq I$ implies that $zI_2 \subseteq I$ and $zI_3 \subseteq I$ for all $z \in I_1$ and thus $I_1I_2 \subseteq I$ and $I_1I_3 \subseteq I$
- If $I_2 \subseteq B_y$ and $I_3 \nsubseteq B_y$ for some $y \in I_1 \setminus I$, then we **claim** that $I_2 \subseteq B_z$ for each $z \in I_1 \setminus I$. Let $z \in I_1 \setminus I$. By theorem (3.4), we conclude that either $B_z \subseteq B_y$ or $B_y \subseteq B_z$. If $B_y \subseteq B_z$, then $I_2 \subseteq B_z$ and we are done. Otherwise assume $B_z \subseteq B_y$. Since $I_1I_2I_3 \subseteq I$, then $I_2I_3 \subseteq B_z$. Since B_z is a prime ideal of S, then we have either $I_2 \subseteq B_z$ or $I_3 \subseteq B_z$. If $I_3 \subseteq B_z$, then we can choose $y = z$ and thus $I_3 \subseteq B_y$, a contradiction. So, $B_y \subseteq B_z$ and thus $I_2 \subseteq B_z$ and $zI_2 \subseteq I$ for all $z \in I_1 \setminus I$ implies that $I_1I_2 \subseteq I$.

- **Case 2**: Suppose that $Rad(I) = P_1 \cap P_2$ where P_1, P_2 are the only distinct prime k-ideals of S that are minimal over I. Since $I_1I_2I_3 \subseteq I \subseteq Rad(I)$, then $I_1I_2I_3 \subseteq P_1$ and $I_1I_2I_3 \subseteq P_1$. Since P_1 and P_2 are prime ideals od S, then we have either $I_1 \subseteq P_1$ or $I_2 \subseteq P_1$ or $I_3 \subseteq P_1$ and either $I_1 \subseteq P_2$ or $I_2 \subseteq P_2$ or $I_3 \subseteq P_2$. Assume $I_1 \subseteq P_1$. Then we consider three cases:

1. If $I_1 \subseteq P_1$ and either $I_2 \subseteq P_2$ or $I_3 \subseteq P_2$, then either $I_1 I_2 \subseteq I$ or $I_1 I_3 \subseteq I$ because $P_1 P_2 \subseteq I$ by theorem (3.5).
2. If $I_1 \subseteq P_1 \cap P_2$ and $I_1 \subseteq I$, then $I_1 I_2 \subseteq I$ and $I_1 I_3 \subseteq I$.
3. If $I_1 \subseteq P_1 \cap P_2$ and $I_1 \nsubseteq I$, then we follow the same argument in case (1) and we are done.

$\square$

CHAPTER 4

On 2-Absorbing Ideals in Special Categories of Semirings

4.1 2-Absorbing Ideals and P-Primal k-Ideals

In this section, we recall the definition of P-primal ideals and introduce the relationship between the 2-absorbing ideals and P-primal ideals of a semiring S.

Definition 4.1 (P-Primal Ideal)**.** Let S be a semiring and I an ideal of S. Then I is said to be P-primal ideal of S if $Z(S/I) = P/I$ for some a prime ideal P.

Theorem 4.1. *Suppose that I is a P-primal k-ideal of a semiring S such that $Rad(I) = P$. Then the following are equivalent:*

(1) I is a 2-absorbing ideal of S.

(2) $P^2 \subseteq I$.

Proof. (1) $\Rightarrow$ (2) Assume I is a 2-absorbing k-ideal of S such that $Rad(I) = P$. Then by theorem (3.3), $P^2 \subseteq I$.

(2) $\Rightarrow$ (1) Suppose I is a P-primary ideal of S such that $P^2 \subseteq I$ and let x, y and $z \in S$ with $xyz \in I$. Since $I \subseteq Rad(I) = P$, then $xyz \in P$. Since P is prime ideal of S, then either $x \in P$ or $yz \in P$. If either $x \in I$ or $yz \in I$, then we are done. Assume that neither $x \in I$ nor $yz \in I$. Since $xyz \in I$, then $xyz + I = (x + I)(yz + I) = I$ implies that $x + I,\ yz + I \in Z(S/I)$. Since I is P-primarl ideal of S, then $x \in P$ and $yz \in P$. Since P is a prime ideal of S, then we conclude either $x,\ y \in P$ or $x,\ z \in P$. Since $P^2 \subseteq I$, then we have either $xy \in I$ or $xz \in I$. Hence, I is a 2-absorbing ideal of S. □

4.2 On 2-Absorbing Ideals of Divided Semidomains

In this section, we study the concepts of divided semidomains and divided ideals in a semiring S. We also investigate the notation of 2-absorbing ideals of a divided semidomain and discuss theorems and examples related to it.

Definition 4.2. Let S be a semiring and P a prime ideal of S. Then P is said to be a divided prime ideal if $P \subset \langle x \rangle$ for every $x \in S \setminus P$.

Remark 4.1. *If P is a divided prime ideal of a semiring S, then either $P \subset \langle x \rangle$ or $\langle x \rangle \subset P$ for every $x \in S$. That means, P is comparable to every principle ideal of S.*

Proof. Assume S is a semiring and P is a divided prime ideal of S. Let $x \in S$. Then either $x \in P$ or $x \in S \setminus P$. If $x \in P$, then $\langle x \rangle \subset P$. If $x \in S \setminus P$, then $P \subset \langle x \rangle$ by the definition of a divided prime ideal. $\square$

Definition 4.3 (Divided Semdomain)**.** A semidomain S is said to be a divided semdomain if every prime ideal of S is a divided prime ideal.

Theorem 4.2. *Suppose that P is nonzero divided prime k-ideal of a semiring S and I is a k-ideal such that $Rad(I) = P$. Then the following statements are equivalent:*

(1) I is a 2-absorbing ideal of S.

(2) I is a P-primarl ideal of S such that $P^2 \subseteq I$.

Proof. (2) $\Rightarrow$ (1). It follows from theorem (4.1).

(1) $\Rightarrow$ (2). Suppose I is a 2-absorbing ideal of S. Since $Rad(I) = P$, then by theorem (3.3) we have $P^2 \subseteq I$. Now, we show the equality $Z(S/I) = P/I$, let $x + I \in P/I$. If $x \in I$, then $x + I = I$ which implies $x + I \in Z(S/I)$. If $x \in P \setminus I$, then $x^2 \in I$ because $P^2 \subseteq I$. Hence, $x^2 + I = (x + I)(x + I) = I$ so $x + I \in Z(S/I)$ and thus $P/I \subseteq Z(S/I)$. To prove the other direction of the equality, let $0 \neq x + I \in Z(S/I)$. Then there exists $0 \neq y + I \in S/I$ such that $(x + I)(y + I) = (xy) + I = I$ implies that $xy \in I$ and $x,\ y \in S \setminus I$. Since P is a prime ideal and $xy \in I \subseteq P$, then we conclude that either $x \in P$ or $y \in P$. Assume $x \notin P$ and $y \in P$. Since P is a divided prime ideal of S, then $P \subset \langle x \rangle$ so there exists $k \in S$ such that $y = xk$ and thus we have $xy = x^2k \in I$. Since I is a 2-absorbing ideal of S and $y = xk \notin I$, then we have $x^2 \in I \subseteq P$. Since P is a prime ideal, then $x \in P$, a contraction. Hence, $x, y \in P$ and $Z(S/I) = P/I$. $\square$

Theorem 4.3. *Suppose that S is a multiplicatively cancellative semiring and P is a divided prime k-ideal of S. Then P^2 is a 2-absorbing ideal of S if P^2 is a k-ideal of S.*

Proof. Let S be a multiplicatively cancellative semiring and P be a divided prime k-ideal of S and suppose P^2 is k-ideal of S. Then by theorem (2.1), we have $Rad(P^2) = Rad(P)$. Now, to show P^2 is a 2-absorbing ideal. It suffices by theorem (4.2) to prove that P^2 is a P-primary ideal of S, i.e., $Z(S/P^2) = P/P^2$. Let $a + P^2 \in P/P^2$. Then $a \in P$ implies that $a^2 \in P^2$. So, $(a + P^2)(a + P^2) = a^2 + P^2 = P^2$. Hence, $a + P^2 \in Z(S/P^2)$ and thus $P/P^2 \subset Z(S/P^2)$. Now, we prove the other direction of the equality let $0 \neq x + P^2 \in Z(S/P)$. Then there exists $0 \neq y + P^2 \in S/P^2$ such that $(x+P^2)(y+P^2) = xy+P^2 = P^2$, which implies $xy \in P^2$ and then there exist $\{p_1, p_2, ... \ p_n\}$ and $\{q_1, q_2, ... \ q_n\}$ in P such that $xy = p_1q_1 + p_2q_2 + ... \ + p_nq_n$. Since $xy \in P^2 \subset P$ and P is a prime ideal of S, then we have either $x \in P$ or $y \in P$. Assume $x \notin P$. Since P is a divided prime ideal of S, then for all $i \in \{1, 2, ... \ n\}$ we have $p_i = xc_i$ where the c_i's are in S and thus $xy = xc_1q_1 + xc_2q_2 + ... \ + xc_nq_n$. Since P is a prime ideal of S and $x \notin P$, then $c_i \in P$ so all c_i's are in P. Since S is a multiplicatively canellative semiring, then we have $y = c_1q_1 + c_2q_2 + ... \ + c_nq_n \in P^2$, a contradiction because $y \notin P^2$. So, $x \in P$ and $x + P^2 \in P/P^2$ and thus $Z(S/P^2) = P/P^2$. Therefore, P^2 is a P-primarl ideal of S and so P is a 2-absorbing ideal of S. □

Theorem 4.4. *Suppose that S is a semiring and nilradical of S ($Nil(S)$) is a prime ideal of S. Let P be a divided prime k-ideal of S such that $Nil(S) \subset P^2$. Then P^2 is 2-absorbing ideal of S if P^2 is a k-ideal of S and $P^2 \subset V(S)$.*

Proof. Let P be a divided prime k-ideal of S. Then by theorem (2.1), we have $Rad(P^2) = P$. To prove P^2 is a 2-absorbing ideal of S it is enough to show that P^2 is P-primary ideal i.e., $Z(S/P^2) = P/P^2$. The first direction of the equality ($P/P^2 \subseteq Z(S/P^2)$) follows from theorem (4.3). Now, let $0 \neq x + P^2$. Then there exists $0 \neq y + P^2 \in S/P^2$ such that $(x + P^2)(y + P^2) = xy + P^2 = P^2$ which implies $xy \in P^2$ and then $xy = p_1q_1 + p_2q_2 + \cdots + p_nq_n$ where the p_i's and q_i's are in P. Since $xy \in P^2 \subset P$ and P is a prime ideal of S, then we have either $x \in P$ or $y \in P$. Assume $x \notin P$. Since P is a divided prime ideal of S, then for all $i \in \{1, 2, \cdots n\}$ we have $p_i = xc_i$ where the c_i's are in S and thus $xy = xc_1q_1 + xc_2q_2 + \cdots + xc_nq_n$. Since P is a prime ideal of S and $x \notin P$, then $c_i \in P$ so all c_i's are in P. Since $P^2 \subset V(S)$, then we conclude that $xy - xc_1q_1 - xc_2q_2 - \cdots - xc_nq_n = x(y - c_1q_1 - c_2q_2 - \cdots - c_nq_n) = 0 \in Nil(S)$. Since $x \notin P$ and $Nil(S) \subset P$, then $x \notin Nil(S)$ and so $(y - c_1q_1 - c_2q_2 - ... - c_nq_n) = z \in Nil(S)$ because $Nil(S)$ is a prime ideal of S. Since $Nil(S) \subset P^2$, then we have that $y = c_1q_1 + c_2q_2 + ... + c_nq_n + z \in P^2$, a contradiction. So, $x \in P$ and $Z(S/P^2) = P/P^2$. Therefore, P^2 is 2-absorbing ideal of S. □

Corollary 4.1. *Suppose S is a semidomain and P is a nonzero divided prime k-ideal. Then P^2 is 2-absorbing ideal of S if P^2 is k-ideal and $P^2 \subset V(S)$.*

Proof. Let S be a semidomain. Then $Nil(S) = 0$ is a prime ideal and hence $Nil(S) \subset P^2$. So, P^2 is a 2-absorbing ideal of S by theorem (4.4) . □

We consider an example of a semidomain S and a prime k-ideal P of S such that P^2 is not a 2-absorbing ideal of S.

Example 4.1. Suppose that $S = \mathbb{N}+4x\mathbb{N}[x]$ where $\mathbb{N}$ is the semiring of integers and x is an indeterminate. Then S is a commutative semiring by example (2.5). To show S has no nonzero zero divisor, let $a, b \in S$

with $ab = 0$. Then there exist $c_1,\ c_2 \in \mathbb{N}$ and $f_1(x),\ f_2(x) \in \mathbb{N}[x]$ such that $a = c_1 + 4xf_1(x)$ and $b = c_2 + 4xf_2(x)$ and then $ab = c_1c_2 + 4x[c_1f_2(x) + c_2f_1(x) + 4xf_1(x)f_2(x)] = 0$. So, $ab = 0$ if either c_1 and $f_1(x)$ are equal to 0 or c_2 and $f_2(x)$ are equal to 0. Hence, either $a = 0$ or $b = 0$ and thus S is a semidomain.

Assume $P = 4x\mathbb{N}[x]$. To show P is a prime ideal of S, Let $y, z \in S$ with $yz \in P$ and $y \notin P$. Then there exist $c_1,\ c_2 \in \mathbb{N}$ and $f_1(x),\ f_2(x) \in \mathbb{N}[x]$ such that $y = c_1+4xf_1(x)$ and $z = c_2+4xf_2(x)$ and then $yz = c_1c_2 + 4x[c_1f_2(x) + c_2f_1(x) + 4xf_1(x)f_2(x)]$. Since $yz \in P$, then $c_1c_2 = 0$ if $c_1 = 0$ then $y \in P$, a contradiction. Therefore, $c_2 = 0$ and then $z \in P$. To prove P is k-ideal, let $a, b \in S$ with $a + b \in P$ and $a \in P$. Then there exists $c \in \mathbb{N}$ and $f_1(x),\ f_2(x),\ f_3(x) \in \mathbb{N}[x]$ such that $a = 4xf_1(x)$ and $b = c + 4xf_2(x)$ and $ab = 4xf_3(x)$. So, $a + b = c + 4x[f_1(x) + f_2(x)] = 4xf_3(x)$ and hence c must be equal to 0.

To show P^2 is not a 2-absorbing ideal we will use theorem (3.6) i.e., for some $z \in P \setminus P^2$ we have B_z is not a prime ideal. Consider $z = 4x^2$ then $z \notin P^2$ and so $z \in P \setminus P^2$. Moreover, $B_z = B_{4x^2} = \{y \in S \mid y(4x^2) \in P^2\} = 4\mathbb{N} + 4x\mathbb{N}[x]$ is not a prime ideal of S. Since $(2+4x)(2+4x) = 4+4x[4+4x] \in B_{4x^2}$ and $2+4x \notin B_{4x^2}$. Hence, P^2 is not 2-absorbing ideal of S.

4.3 On 2-Absorbing Ideals of Valuation Semirings

In this section, we give the definition of valuation semiring. We also introduce the conection between a divided semidomain and valuation semiring and we study relevance between 2-absorbing ideals and P-primarl ideals of valuation semiring.

First let us consider the connotation of valuation maps of semirings with values in tomonoid.

Definition 4.4. [15] An M-valuation f on a semiring S is a map $f : S \longrightarrow M_\infty$ such that the following conditions hold:

(1) $(M_\infty, +, 0, \leq)$ is tomonoid with the largest element $+\infty$, which has gained from the tomonoid $(M, +, 0, \leq)$ with no largest element.

(2) $f(ab) = f(a) + f(b)$ for all $a, b \in S$.

(3) $f(a + b) \geq \min\{f(a), f(b)\}$ for all $a, b \in S$.

(4) $f(1) = 0$ and $f(0) = +\infty$.

Now let us give an example of an M-valuation map f on a semiring S.

Example 4.2. Suppose that S is a semiring with no nonzero zero divisors ($Z(S) = \{0\}$). Then

$$f(s) = \begin{cases} 0, & s \in S \setminus \{0\} \\ +\infty, & s = 0 \end{cases}$$

is an M-valuation f on S where $M = \{0\}$. To show this we check the four previous conditions of definition (4.4).

- M_∞ is tomonoid with greatest element $+\infty$.

- Let $a, b \in S$. If either a or b equal to 0, then $ab = 0$ and hence $+\infty = f(ab) = f(a) + f(b)$. Now, assume neither a nor b equal to 0. Since S has no nonzero zero divisors, then $ab \neq 0$ and so $0 = f(ab) = f(a) + f(b)$. Hence, $f(ab) = f(a) + f(b)$ for all $a, b \in S$.

- Let $a, b \in S$. If a and b equal to 0, then $f(a+b) = +\infty = \min\{f(a), f(b)\}$. If $a = 0$ and $b \neq 0$, then $a + b = b$ and $f(a+b) = 0 = \min\{f(a), f(b)\}$. Now, assume neither a nor b equal to 0 if $a+b = 0$ then $f(a+b) = +\infty \geq \min\{f(a), f(b)\} = 0$. If $a + b \neq 0$, then $f(a+b) = 0 = \min\{f(a), f(b)\}$. So, in either any cases we have $f(a+b) \geq \min\{f(a), f(b)\}$.

- $f(0) = +\infty$ and $f(1) = 0$ from the assumption.

Definition 4.5. Let S be a semiring and $S_f = \{s \in S,\ f(s) \geq 0\}$. Then S_f is said to be a **F-semiring with respect to the triple** (S, f, M) if there exists an M-valuaion f on S.

Definition 4.6 (Valuation Semiring)**.** Let S be a semiring. Then S is said to be a valuation semiring if there exists an M-valuation f on K, where K is a semifield and f is a surjective map and $S = K_f = \{s \in K,\ f(s) \geq 0\}$.

Theorem 4.5. *Let S be a multiplicatively cancellative semiring. Then S is a divided semidomain if it's a valuation semiring.*

Proof. Let P be a prime ideal of a multiplicatively cancellative valuation semiring S and $x \in S \setminus P$ and $y \in P$. Then by [15] we have either $\langle x \rangle \subseteq \langle y \rangle$ or $\langle y \rangle \subseteq \langle x \rangle$. If $\langle y \rangle \subseteq \langle x \rangle$, then $P \subseteq \langle x \rangle$ and we are done. If $\langle x \rangle \subseteq \langle y \rangle$, then there exists $s \in S$ such that $x = ys$. Since $y \in P$, then $x = ys \in P$, a contradiction. Hence, $P \subset \langle x \rangle$ for all $x \in S \setminus P$ and thus S is a divided semidomain. $\square$

Theorem 4.6. *Suppose that S is a multiplicatively cancellative valuation semiring and I is a nonzero proper k-ideal of S such that $Rad(I) = P$. Then I is a 2-absorbing ideal of S if and only if I is P-primarl ideal of S such that $P^2 \subseteq I$.*

Proof. ($\Rightarrow$) Assume I is a 2-absorbing ideal of S. Then $Rad(I) = P$ is a prime k-ideal of S. Since S is a multiplicatively cancellative valuation, then by theorem (4.5) S is a divided domain and so P is a divided prime ideal of S. By theorem (4.2), I is P-primarl ideal of S such that $P^2 \subseteq I$.

($\Leftarrow$) Assume I is a P-primal ideal of S such that $P^2 \subseteq I$. Since S is a divided semidomain, then by theorem (4.2) I is 2-absorbing ideal of S. □

Theorem 4.7. *Suppose that S is a multiplicatively cancellative valuation semiring and I is a nonzero proper k-ideal of S. Let $Rad(I) = P$ and P^2 is a k-ideal of S. Then I is a 2-absorbing of S if $I = P$ or $I = P^2$.*

Proof. Suppose that either $I = P$ or $I = P^2$ where $P = Rad(I)$. Then P is a prime k-ideal of S. If $I = P$, then I is a 2-absorbing ideal of S. Now assume $I = P^2$. Since S is a multiplicatively cancellative valuation semiring, then by theorem (4.5) S is a divided semidomain and so P is a divided prime ideal of S. By theorem (4.3), we have P^2 is a 2-absorbing ideal of S. □

The following is an example of a semidomain S and a prime k-ideal P of S such that P^2 is not a P-primarl ideal of S, but P^2 is a 2-absorbing ideal of S.

Example 4.3. Assume that $S = \mathbb{Z} + 3x\mathbb{Z}[x]$ where $\mathbb{Z}$ is the semiring of integer numbers and x is an indeterminate. Then S is a commutative semiring by example (2.5). To show that S has no nonzero zero divisor, let $a, b \in S$ with $ab = 0$. Then there exist $c_1, c_2 \in \mathbb{N}$ and $f_1(x), f_2(x) \in \mathbb{N}[x]$ such that $a = c_1 + 3xf_1(x)$ and $b = c_2 + 3xf_2(x)$ and then $ab = c_1c_2 + 3x[c_1f_1(x) + c_2f_1(x) + 3xf_1(x)f_2(x)] = 0$. So, $ab = 0$ if either c_1

and $f_1(x)$ are equal to 0 or c_2 and $f_2(x)$ are equal to 0. Hence, either $a = 0$ or $b = 0$ and thus S is a semidomain.

Suppose $P = 3x\mathbb{Z}[x]$. To show P is a prime k-ideal of S, let $y, z \in S$ with $yz \in P$ and $y \notin P$. Then there exist $c_1, c_2 \in \mathbb{N}$ and $f_1(x), f_2(x) \in \mathbb{N}[x]$ such that $y = c_1 + 3xf_1(x)$ and $z = c_2 + 3xf_2(x)$ and then $yz = c_1c_2 + 3x[c_1f_1(x) + c_2f_1(x) + 3xf_1(x)f_2(x)]$. Since $yz \in P$, then $c_1c_2 = 0$ if $c_1 = 0$ then $y \in P$, a contradiction. Therefore, $c_2 = 0$ and then $z \in P$. To prove P is k-ideal, let $a,\ b \in S$ with $a + b \in P$ and $a \in P$. Then there exists $c \in \mathbb{N}$ and $f_1(x),\ f_2(x),\ f_3(x) \in \mathbb{N}[x]$ such that $a = 3xf_1(x)$ and $b = c + 3xf_2(x)$ and $ab = 3xf_3(x)$. So, $a + b = c + 3x[f_1(x) + f_2(x)] = 3xf_3(x)$ and hence c must be equal to 0.

P^2 is not P-primarl ideal of S since if we take $a = 3 + 3x$ and $b = 3x^2$, then a and $b \notin P^2$. Consider $ab = (3 + 3x)3x^2 = (3x)(3x) + (3x)(3x^2)$. Then $ab \in P^2$ and thus a and $b \in Z(S/P^2)$, but $a \notin P$. Therefore, $Z_S/P^2 \neq P/P_2$.

To show P^2 is a 2-absorbing ideal of S we will use theorem (3.6), let $f \in P \setminus P^2$. Then we have either $B_f = \{y \in S \mid yf \in P^2\} = P$ or $B_f = 3\mathbb{Z} + 3x\mathbb{Z}[x]$ and in either two cases B_f is a prime ideal.

Conclusion

In this thesis we recalled some of algebraic structures in semiring theory and gave some examples related to it. We studied the concept of 2-absorbing ideal in commutative semiring and illustrated it with many examples and introduced advanced theorems, also we studied this concept in particular classes of a semiring.

Future Work

In future we hope to study the concept of 2-absorbing ideal in prufer semidomain and Dedekind semidomain. Also we wish to study another generalization of prime ideals in commutative semirings, for example n-absorbing ideals and primary ideals.

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